An axisymmetric DtN Finite Element Method for problems in geophysics

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Overview

1. Introduction
2. Mathematical model & DtN FEM formulation
3. Coupling DtN map with FEM
4. Numerical results & validation
5. Conclusions and future work
1 Introduction

2 Mathematical model & DtN FEM formulation

3 Coupling DtN map with FEM

4 Numerical results & validation

5 Conclusions and future work
DtN Finite Element Method (standard version)

Exterior domain $\Omega$ (complement of a bounded obstacle) and some boundary-value problem (BVP) formulated in $\Omega$
Introduce an artificial boundary $\Gamma$ dividing $\Omega$ into an internal bounded domain $\Omega^i$ and an external unbounded domain $\Omega^e$.
DtN Finite Element Method (standard version)

By solving analytically the BVP in $\Omega^e$, deduce a relation on $\Gamma$ involving the unknown function $u$ and its derivatives.
The Dirichlet-to-Neumann (DtN) map provides such a relation as an explicit closed-form expression (for most standard BVPs)
Use the DtN map relation as boundary condition on $\Gamma$ and solve numerically by FEM the resulting BVP in $\Omega^i$.
From a geometrical point of view the earth is better described by a semi-infinite domain than by an exterior domain.
Applying DtN FEM to problems in geophysics

A linear elastic model is used to describe the mechanical behaviour of the solid earth
Applying DtN FEM to problems in geophysics

For the BVP of elastostatics in a semi-infinite domain, an explicit closed-form expression for the DtN map is not possible.
Applying DtN FEM to problems in geophysics

To apply a DtN FEM procedure to problems in geophysics, some suitable approximation of the DtN map needs to be considered.
Approach considered in this work

- A DtN FEM method is developed for BVPs of linear elastostatics in a semi-infinite domain
- The semi-infinite domain has rotational symmetry about the vertical axis (axisymmetry)
- An isotropic, homogeneous, elastic solid medium occupies the semi-infinite domain
- Use of a semi-analytical method to solve the BVP in the unbounded external domain
- Numerical approximation of the terms involving the DtN map of the FEM formulation
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Mathematical model in $\Omega$

Find $u : \Omega \longrightarrow \mathbb{R}^2$ such that

\[
\begin{align*}
\text{div } \sigma(u) &= 0 \quad \text{in } \Omega \\
\sigma(u)\hat{k} &= 0 \quad \text{on } \Gamma_\infty \\
\sigma(u)\hat{n} &= f \quad \text{on } \Gamma_p \\
\sigma(u)\hat{n} \cdot \hat{k} &= u \cdot \hat{n} = 0 \quad \text{on } \Gamma_s \\
|u| &\sim \frac{1}{|x|} \quad \text{as } |x| \to \infty
\end{align*}
\]

$u, \sigma$ Displacement field & stress tensor

$\sigma(u) = \lambda (\text{div } u) I + 2\mu (\nabla u + \nabla u^T)$

Isotropic Hooke’s law

$E. \text{ Godoy & M. Durán INGMAT}$

An axisymmetric DtN Finite Element Method for problems in geophysics
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|\mathbf{u}| &\sim \frac{1}{|\mathbf{x}|} \quad \text{as } |\mathbf{x}| \rightarrow \infty
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$\mathbf{\sigma}(\mathbf{u}) = \lambda(\text{div } \mathbf{u}) \mathbf{I} + 2\mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$

Isotropic Hooke's law
Find $u : \Omega^i \rightarrow \mathbb{R}^2$ such that

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\sigma(u)\hat{n} &= f & \text{on } \Gamma_p \\
\sigma(u)\hat{\rho} \cdot \hat{k} &= u \cdot \hat{\rho} = 0 & \text{on } \Gamma^i_s \\
\sigma(u)\hat{n} &= -\mathcal{M}u & \text{on } \Gamma_R
\end{align*}

where $\mathcal{M}$ denotes the DtN map (assumed for the moment to be known)
Weak formulation

The function space of *physically admissible* displacements is

\[ V = \{ \mathbf{v} = (v_\rho, v_z) \in [H^1(\Omega^i)]^2 : v_\rho|_{\Gamma^i_s} = 0 \} \]

The weak formulation is

\[
(WF) \begin{cases} 
\text{Find } u \in V \text{ such that} \\
a(u, v) + b(u, v) = (f, v)_{\Gamma_p} & \forall v \in V 
\end{cases}
\]

with

\[
a(u, v) = \int_{\Omega^i} \left[ \sigma_\rho(u) \frac{\partial v_\rho}{\partial \rho} + \sigma_\theta(u) \frac{v_\rho}{\rho} + \sigma_z(u) \frac{\partial v_z}{\partial z} + \sigma_{\rho z}(u) \left( \frac{\partial v_\rho}{\partial z} + \frac{\partial v_z}{\partial \rho} \right) \right] \, dx
\]

\[
b(u, v) = \int_{\Gamma^R} \mathbf{M} \mathbf{u} \cdot \mathbf{v} \, ds, \quad (f, v)_{\Gamma_p} = \int_{\Gamma_p} f \cdot v \, ds
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**Weak formulation**

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\]
P1-finite element discretisation

$\mathcal{T}_h$ a triangular mesh of size $h > 0$ such that $\overline{\Omega^i} = \bigcup_{T \in \mathcal{T}_h} \overline{T}$

Define $\mathcal{V}^h = \{ \mathbf{v}^h \in \mathcal{V} : \mathbf{v}^h \in [C^0(\Omega^i)]^2, \mathbf{v}^h|_T \in [P_1(T)]^2 \ \forall \ T \in \mathcal{T}^h \}$

The discrete weak formulation is

$$\text{(WF}_h) \left\{ \begin{array}{l}
\text{Find } u^h \in \mathcal{V}^h \text{ such that }
\end{array} \right.

a(u^h, v^h) + b(u^h, v^h) = (f, v^h)_{\Gamma_p} \quad \forall \ v^h \in \mathcal{V}^h$$

$I$ the set of nodes in $\mathcal{T}_h$, $\{\psi_j\}_{j \in I}$ the usual nodal shape functions, $I_s \subset I$ the set of nodes lying on $\Gamma^i_s$

The solution $u^h$ is approximated as

$$u^h(x) = \sum_{j \in I \setminus I_s} \alpha_{\rho,j} \psi_j(x) \hat{\rho} + \sum_{j \in I} \alpha_{z,j} \psi_j(x) \hat{k}$$
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\( \mathcal{I} \) the set of nodes in \( \mathcal{T}^h \), \( \{\psi_j\}_{j \in \mathcal{I}} \) the usual nodal shape functions, 
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\]
Finite element matrix form

Substituting $v^h$ by $\psi_j \hat{\rho}$ and $\psi_j k$ in $(WF^h)$ leads to the finite element matrix form of the problem:

$$
\begin{bmatrix}
K^a_{\rho\rho} + K^b_{\rho\rho} & K^a_{\rho z} + K^b_{\rho z} \\
K^a_{z\rho} + K^b_{z\rho} & K^a_{zz} + K^b_{zz}
\end{bmatrix}
\begin{bmatrix}
\alpha_{\rho} \\
\alpha_{z}
\end{bmatrix} =
\begin{bmatrix}
F_{\rho} \\
F_{z}
\end{bmatrix}
$$

The following matrices involving the DtN map need to be approximated:

$$
[K^b_{\rho\rho}]_{ij} = \int_{\Gamma_R} \psi_i \hat{\rho} \cdot \mathcal{M} \psi_j \hat{\rho} \, ds, \quad [K^b_{\rho z}]_{ij} = \int_{\Gamma_R} \psi_i \hat{\rho} \cdot \mathcal{M} \psi_j \hat{k} \, ds,
$$

$$
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The following matrices involving the DtN map need to be approximated

$$[K^b_{\rho\rho}]_{ij} = \int_{\Gamma_R} \psi_i \hat{\rho} \cdot M \psi_j \hat{\rho} \, ds,$$
$$[K^b_{\rho z}]_{ij} = \int_{\Gamma_R} \psi_i \hat{\rho} \cdot M \psi_j \hat{k} \, ds,$$
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DtN map definition

\[ M : [H^{1/2}(\Gamma_R)]^2 \rightarrow [H^{-1/2}(\Gamma_R)]^2 \]

\[ v \rightarrow Mv = -\sigma(u)\hat{r}\big|_{\Gamma_R} \]

where \( u \) is the solution of:

Find \( u : \Omega^e \rightarrow \mathbb{R}^2 \) such that

\[ \text{div} \, \sigma(u) = 0 \quad \text{in} \; \Omega^e \]

\[ \sigma(u)\hat{k} = 0 \quad \text{on} \; \Gamma^e_\infty \]

\[ u = v \quad \text{on} \; \Gamma_R \]

\[ \sigma(u)\hat{\rho} \cdot \hat{k} = u \cdot \hat{\rho} = 0 \quad \text{on} \; \Gamma^e_s \]

\[ |u| \sim \frac{1}{r} \quad \text{as} \; r \rightarrow \infty \]

Only cases \( v = \psi_j\hat{\rho}, \psi_j\hat{k} \) are required!
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\[ \mathcal{M} : [H^{1/2}(\Gamma_R)]^2 \rightarrow [H^{-1/2}(\Gamma_R)]^2 \]
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Only cases \( v = \psi_j \hat{\rho}, \psi_j \hat{k} \) are required!
Semi-analytical solution in $\Omega^e$

Suitable application of separation of variables in $(r, \phi)$ yields

\[
\mathbf{u}(r, \phi) = \sum_{n=0}^{\infty} A_n \left( \frac{R}{r} \right)^{2n+2} \mathbf{w}_n^{(A)}(\phi) + \sum_{n=-1}^{\infty} B_n \left( \frac{R}{r} \right)^{2n+3} \mathbf{w}_n^{(B)}(\phi)
\]

\[
\sigma(r, \phi) = \frac{1}{R} \left[ \sum_{n=0}^{\infty} A_n \left( \frac{R}{r} \right)^{2n+3} \mathbf{\tau}_n^{(A)}(\phi) + \sum_{n=-1}^{\infty} B_n \left( \frac{R}{r} \right)^{2n+4} \mathbf{\tau}_n^{(B)}(\phi) \right]
\]

Functions $\mathbf{w}_n^{(A)}$, $\mathbf{w}_n^{(B)}$, $\mathbf{\tau}_n^{(A)}$, $\mathbf{\tau}_n^{(B)}$ are known

Coefficients $A_n$, $B_n$ are to be determined so that the BC on $\Gamma_R$ holds

This is not possible analytically, so it is do it numerically by minimisation of some quadratic energy functionals appropriately defined

To do so the infinite series are first truncated at a finite order $N$
Semi-analytical solution in $\Omega^e$

Suitable application of separation of variables in $(r, \phi)$ yields

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Functions $w_n^A, w_n^B, \tau_n^A, \tau_n^B$ are known

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Suitable application of separation of variables in $(r, \phi)$ yields

$$u(r, \phi) = \sum_{n=0}^{N} A_n \left( \frac{R}{r} \right)^{2n+2} w_n^A(\phi) + \sum_{n=-1}^{N} B_n \left( \frac{R}{r} \right)^{2n+3} w_n^B(\phi)$$

$$\sigma(r, \phi) = \frac{1}{R} \left[ \sum_{n=0}^{N} A_n \left( \frac{R}{r} \right)^{2n+3} \tau_n^A(\phi) + \sum_{n=-1}^{N} B_n \left( \frac{R}{r} \right)^{2n+4} \tau_n^B(\phi) \right]$$

Functions $w_n^A, w_n^B, \tau_n^A, \tau_n^B$ are known.

Coefficients $A_n, B_n$ are to be determined so that the BC on $\Gamma_R$ holds.

This is not possible analytically, so it is done numerically by minimisation of some quadratic energy functionals appropriately defined.

To do so the infinite series are first truncated at a finite order $N$. 

E. Godoy & M. Durán

An axisymmetric DtN Finite Element Method for problems in geophysics
Quadratic energy functionals

Define the quadratic energy functionals $J$ as

$$J(u) = -\frac{1}{2R^2} \int_{\Gamma_R} \sigma(u) \hat{r} \cdot u \, ds + \frac{1}{R^2} \int_{\Gamma_R} \sigma(u) \hat{r} \cdot v \, ds$$

with $v = \psi_j \hat{\rho}, \psi_j \hat{k}$, for each node $j$ of the mesh lying on $\Gamma_R$.

Substituting $u, \sigma$ in $J$ and expanding yields a matrix form

$$J(x) = \frac{1}{2} x^T Q x - x^T c$$

with

$$Q = \begin{bmatrix} Q^{(AA)} & Q^{(AB)} \\ Q^{(BA)} & Q^{(BB)} \end{bmatrix}, \quad c = \begin{bmatrix} c^{(A)} \\ c^{(B)} \end{bmatrix}, \quad x = \begin{bmatrix} x^{(A)} \\ x^{(B)} \end{bmatrix}$$

$$x^{(A)} = (A_0 \ A_1 \ \ldots \ \ A_N)^T, \quad x^{(B)} = (B_{-1} \ B_0 \ \ldots \ \ B_N)^T$$
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Structure of matrices

\( Q^{(AA)} \) and \( Q^{(BB)} \) are symmetric and positive definite matrices

\( Q^{(BA)} = [Q^{(AB)}]^T \)

Therefore \( Q \) is a symmetric and positive definite matrix
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Therefore, \( Q \) is a symmetric and positive definite matrix.
Minimising the quadratic energy functionals

The minimisation of \( J \) yields

\[
\min_x \left[ \frac{1}{2} x^T Q x - x^T c \right] \quad \Leftrightarrow \quad \nabla_x \left[ \frac{1}{2} x^T Q x - x^T c \right] = 0 \quad \Leftrightarrow \quad Q x = c
\]

which corresponds to a linear system of equations for \( A_n, B_n \).

It is solved by using the Schur-Banachiewicz blockwise inversion formula:

\[
x^{(A)} = \left( [Q^{(AA)}]^{-1} + [Q^{(AA)}]^{-1} Q^{(AB)} \tilde{Q}^{(BB)} [Q^{(BB)}]^{-1} [Q^{(AB)}] [Q^{(AA)}]^{-1} \right) c^{(A)}
\]

\[
- [Q^{(AA)}]^{-1} Q^{(AB)} \tilde{Q}^{(BB)} [Q^{(BB)}]^{-1} c^{(B)}
\]

\[
x^{(B)} = - \tilde{Q}^{(BB)} [Q^{(AB)}] T [Q^{(AA)}]^{-1} c^{(A)} + \tilde{Q}^{(BB)} c^{(B)}
\]

where \( \tilde{Q}^{(BB)} = Q^{(BB)} - [Q^{(AB)}] T [Q^{(AA)}]^{-1} Q^{(AB)} \) is the Schur complement of \( Q^{(BB)} \) in \( Q \).
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where $\tilde{Q}^{BB} = Q^{BB} - [Q^{AB}]^T [Q^{AA}]^{-1} Q^{AB}$ is the Schur complement of $Q^{BB}$ in $Q$
Algorithm to compute the coefficients

To evaluate the Schur-Banachiewicz blockwise inversion formula it is required to invert two matrices:

- The symmetric tridiagonal matrix $Q^{(AA)}$: It is inverted efficiently using the Thomas algorithm for tridiagonal systems.
- The positive definite symmetric full matrix $\tilde{Q}^{(BB)}$: It is inverted efficiently with the aid of its Cholesky factorisation.

The evaluation of the formula yields approximate values of coefficients $A_0, A_1, \ldots, A_N$ and $B_{-1}, B_0, B_1, \ldots, B_N$.

Substituting them in the expression of $\sigma$ as truncated series leads to approximations of the desired terms $M\psi_j \hat{\rho}$, $M\psi_j \hat{z}$. 
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E. Godoy & M. Durán
An axisymmetric DtN Finite Element Method for problems in geophysics
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1. Introduction

2. Mathematical model & DtN FEM formulation

3. Coupling DtN map with FEM

4. Numerical results & validation

5. Conclusions and future work

E. Godoy & M. Durán

An axisymmetric DtN Finite Element Method for problems in geophysics
Benchmark problem

Domain $\Omega$ subject to gravity (lithostatic stress problem):

$$f = -\sigma(u_g)\hat{r} \big|_{\Gamma_p}, \quad u_g(z) = -\frac{\rho g z^2}{2(\lambda + 2\mu)} \hat{k}$$

$\rho$: Mass density of the elastic solid  $g$: Acceleration of gravity

Perturbed boundary $\Gamma_p$ is a half of a sphere of radius $a = 600$ m

Using techniques analogous to those just described, a semi-analytical solution to the BVP in $\Omega$ can be calculated

The DtN FEM technique is applied to the same BVP and both solutions are compared for different FEM meshes
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\[ R = 900 \text{ m} \quad h = 72 \text{ m} \]

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Some numerical results

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Some numerical results

$R = 1000 \text{ m}$  \hspace{1cm} $h = 72 \text{ m}$
Numerical validation

RELATIVE ERROR AS A FUNCTION OF MESH SIZE

- $R=1000$
- $R=900$
- $R=800$

relative error (%)

mesh size (m)

E. Godoy & M. Durán

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E. Godoy & M. Durán

An axisymmetric DtN Finite Element Method for problems in geophysics
Conclusions

- A DtN finite element method for BVPs of linear elastostatics in axisymmetric semi-infinite domains was presented.
- The exterior problem is solved by a semi-analytical method that combines analytical solution and numerical minimisation of quadratic energy functionals.
- The DtN map is numerically approximated in the terms of the FEM formulation where it appears.
- The method was numerically validated by comparing the yielded results with a semi-analytical benchmark.
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E. Godoy & M. Durán
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Perspectives for future work

- Extension to full 3D elastostatics
- Extension to time-harmonic elastodynamics
- Extension to time-dependent elastodynamics
- Application to a real problem in geophysics
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Thanks for your attention!

Any question?