ANNUAL INTERNATIONAL CONFERENCE
DAYS ON DIFFRACTION

An impedance boundary condition in elastodynamics and existence of surface waves

Eduardo Godoy[1] Mario Durán[2] Jean-Claude Nédélec[3]

[1]Ingenieros Matemáticos Consultores Asociados S.A. (INGMAT), Santiago, Chile.

[2]Facultad de Ingeniería, Pontificia Universidad Católica de Chile, Santiago, Chile.

[3]Centre de Mathématiques Appliquées, École Polytechnique, Palaiseau, France.

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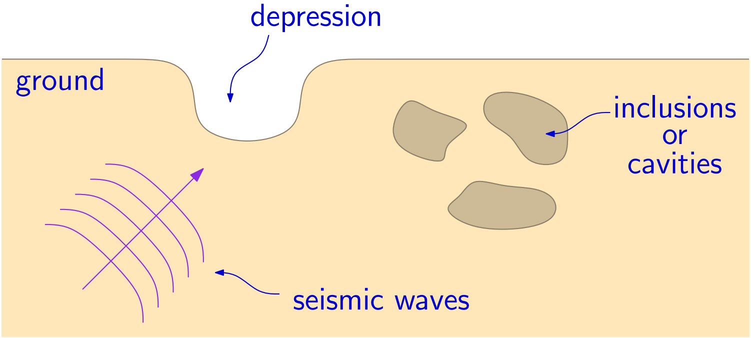
Contents

Motivation: Seismic response of geological structures



Localised amplification of ground motion during earthquakes

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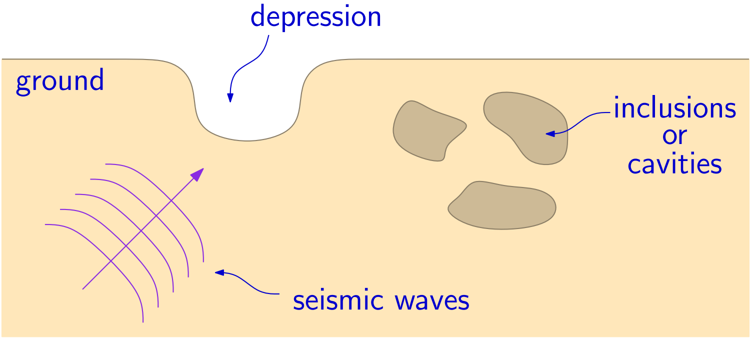

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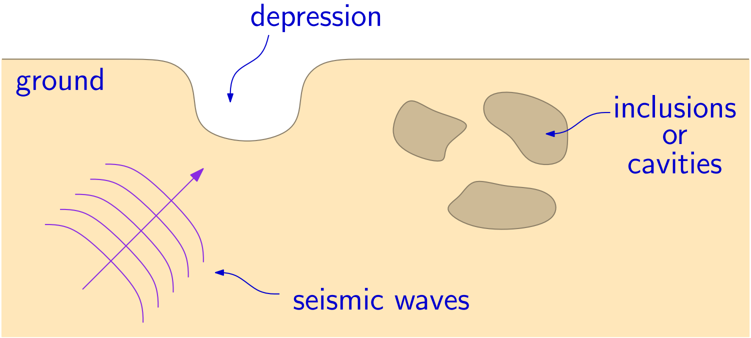

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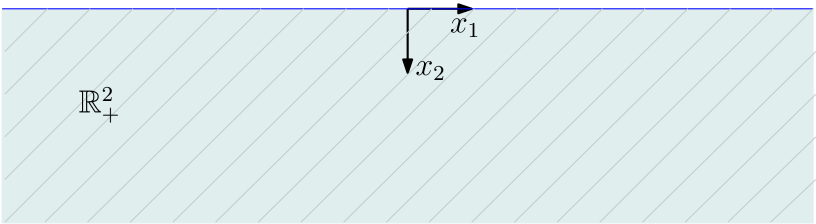

Time-harmonic elastic wave equation: $\sigma_{ij,j} + \rho\,\omega^2 u_i = 0$

Hooke's isotropic law: $\sigma_{ij} = \lambda u_{k,k} + \mu(u_{i,j} + u_{j,i})$

Longitudinal (L) and transverse (T) wave numbers:

$$k_L = \omega\sqrt{\frac{\rho}{\lambda + 2\mu}}, \qquad k_T = \omega\sqrt{\frac{\rho}{\mu}}$$

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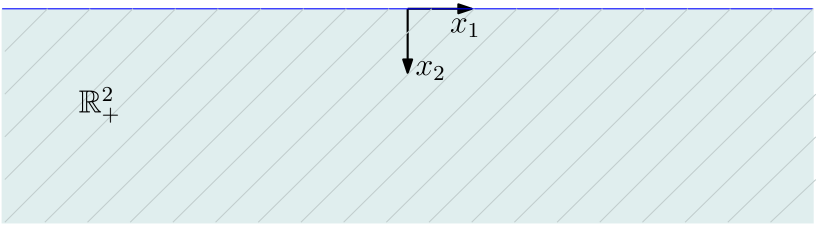

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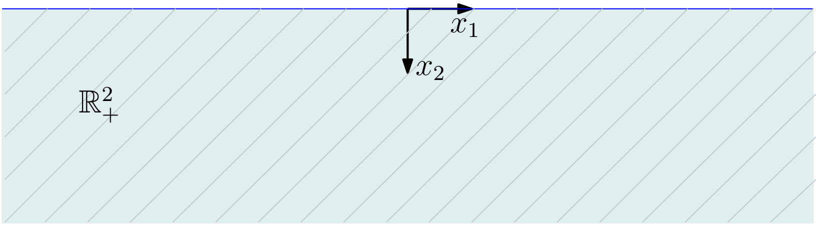

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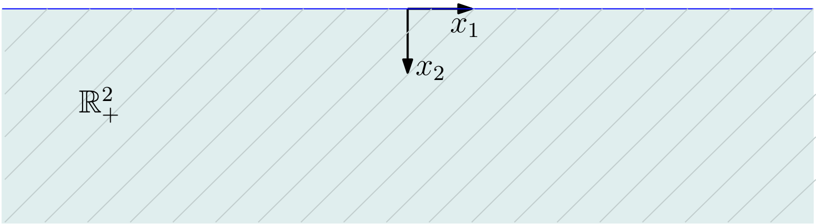

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1. Introduction
2. Boundary conditions
3. Surface waves analysis
4. Conclusions

Traction-free boundary conditions

Given any surface with outward unit normal $\vec{n} = (n_1, n_2)$, the vector quantity

$$t_i = \sigma_{ij} n_j$$

represents the *traction* (force per unit area) on that surface.

A traction-free surface has associated the boundary condition

$$t_i = 0$$

In seismic applications it is usual to assume traction-free surfaces.

The traction-free boundary condition for the half-plane $\mathbb{R}^2_+$ is expressed by

$$\begin{aligned} \sigma_{12} &= 0 \\ \sigma_{22} &= 0 \end{aligned} \qquad \text{on } \{x_2 = 0\}$$

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Impedance boundary conditions

Impedance-like (or Robin) boundary conditions are currently used in fields of physics like acoustics and electromagnetism. However, they are not very well-known in elasticity.

Tiersten (1969) obtained impedance boundary conditions to simulate the influence of a very thin layer on an elastic half-space for low frequencies

$$\left.\begin{aligned} \sigma_{12} &= -4h\,\frac{\mu'(\lambda'+\mu')}{\lambda'+2\mu'}\,u_{1,11} + h\rho'\ddot{u}_1 \\ \sigma_{22} &= h\rho'\ddot{u}_2 \end{aligned}\right\} \quad \text{on } \{x_2 = 0\}$$

where h is the thickness of the layer and the primed constants correspond to the parameters of the layer.

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Bövik (1996) improved Tiersten's boundary conditions by applying a perturbation technique in h [$O(h)$-theory]

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Godoy, Durán and Nédélec (2012) considered the following impedance boundary conditions:

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where Z is an impedance parameter. If $Z = 0$ we retrieve traction-free boundary conditions.

Equation $\sigma_{12} + \omega Z u_1 = 0$ establishes a proportional relation between shear stresses and tangential displacements.

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Surface wave solutions

We look for nonzero solutions (u_1, u_2) of

$$\begin{aligned} \sigma_{ij,j} + \rho\,\omega^2 u_i &= 0 && \text{in } \mathbb{R}^2_+ \\ \sigma_{i2} + \delta_{i1}\omega Z u_i &= 0 && \text{on } \{x_2 = 0\} \end{aligned}$$

that correspond to surface waves, i.e., they have oscillatory behaviour in x_1 and decay exponentially as $x_2 \to +\infty$.

Such surfaces waves are associated with a *surface wave number* k satisfying the following *secular equation*

$$(2k^2 - k_T^2)^2 - 4k^2\sqrt{k^2 - k_L^2}\sqrt{k^2 - k_T^2} + \frac{\omega Z}{\mu}k_T^2\sqrt{k^2 - k_T^2} = 0$$

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Secular equation

The secular equation is expressed in terms of the slowness $s = k/\omega$:

$$\left(2s^2 - s_T^2\right)^2 - 4s^2\sqrt{s^2 - s_L^2}\sqrt{s^2 - s_T^2} + \frac{Z}{\mu}s_T^2\sqrt{s^2 - s_T^2} = 0$$

with $s_L = \sqrt{\rho/(\lambda + 2\mu)}$ and $s_T = \sqrt{\rho/\mu}$ $(2s_L^2 < s_T^2)$.

Standard secular equation for a traction-free surface $(Z = 0)$:

$$\left(2s^2 - s_T^2\right)^2 - 4s^2\sqrt{s^2 - s_L^2}\sqrt{s^2 - s_T^2} = 0$$

which has a unique solution $s_R > s_T$ associated with the well-known *Rayleigh wave*.

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Existence and uniqueness of the Rayleigh wave

Proposition 1

For each impedance $Z \in \mathbb{R}$ the secular equation has a unique solution in the range $s > s_T$. It corresponds to the Rayleigh wave slowness and is strictly increasing in Z.

SKETCH OF THE PROOF: Impedance Z worked out as a function of s:

$$Z(s) = -\frac{\mu}{s_T^2} \frac{\left(2s^2 - s_T^2\right)^2 - 4s^2\sqrt{s^2 - s_L^2}\sqrt{s^2 - s_T^2}}{\sqrt{s^2 - s_T^2}}$$

$Z : (s_T, +\infty) \longrightarrow \mathbb{R}$ is an onto function, since it is continuous and it can be proven that $Z(s)$ satisfies

$$\lim_{s \searrow s_T} Z(s) = -\infty, \qquad \lim_{s \to +\infty} Z(s) = +\infty$$

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Additionally, it is proven that

$$Z'(s) > 0 \qquad \forall s > s_T$$

so Z is an strictly increasing function, and in particular it is one–to–one.

Consequently, $Z : (s_T, +\infty) \to \mathbb{R}$ is a bijective function, and so is its inverse.

We conclude that for each $Z \in \mathbb{R}$ there is a unique solution $s = s_R$ to the secular equation such that $s_R > s_T$. It corresponds to the slowness of the Rayleigh wave.

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Additional surface wave

Proposition 2

If the impedance Z takes the positive value $Z^* \equiv 2\mu\sqrt{s_T^2/2 - s_L^2}$, then the secular equation has one real solution in the range $s_L < s < s_T$. This solution is given by $s^* \equiv s_T/\sqrt{2}$ and corresponds to the slowness of an additional surface wave

SKETCH OF THE PROOF: Assuming $s_L < s < s_T$, the secular equation becomes

$$(s_T^2 - 2s^2)^2 - \mathrm{i}\left(4s^2\sqrt{s^2 - s_L^2} - \frac{1}{\mu}s_T^2 Z\right)\sqrt{s_T^2 - s^2} = 0$$

The first term of this relation is strictly real and the second is purely imaginary.

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Taking real part:

$$(s_T^2 - 2s^2)^2 = 0 \quad \Leftrightarrow \quad s = s^* = \frac{s_T}{\sqrt{2}} \qquad (s_L < s^* < s_T)$$

and taking imaginary part:

$$\frac{s_T^3}{\sqrt{2}}\left(2\sqrt{\frac{s_T^2}{2} - s_L^2} - \frac{1}{\mu}Z\right) = 0 \quad \Leftrightarrow \quad Z = Z^* = 2\mu\sqrt{\frac{s_T^2}{2} - s_L^2}$$

We thus conclude that if $Z = Z^*$, then $s = s^*$ is solution to the secular equation. It corresponds to the slowness of an additional surface wave, which is faster than the transverse wave and slower than the longitudinal wave.

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Conclusions

- A particular impedance-like boundary condition in elastodynamics was considered.
- The secular equation for this boundary condition was obtained explicitly.
- The Rayleigh surface wave was generalised as a function of the impedance parameter.
- An additional surface wave arises if the impedance takes a particular value, whose velocity lies between those of the longitudinal wave and the transverse wave.

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Thanks for your attention!