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On the existence of surface waves in an elastic half-space with impedance boundary conditions

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ABSTRACT

In this work, the problem of surface waves in an isotropic elastic half-space with impedance boundary conditions is investigated. It is assumed that the boundary is free of normal traction and the shear traction varies linearly with the tangential component of displacement multiplied by the frequency, where the impedance corresponds to the constant of proportionality. The standard traction-free boundary conditions are then retrieved for zero impedance. The secular equation for surface waves with impedance boundary conditions is derived in explicit form. The existence and uniqueness of the Rayleigh wave is properly established, and it is found that its velocity varies with the impedance. Moreover, we prove that an additional surface wave exists in a particular case, whose velocity lies between those of the longitudinal and the transverse waves. Numerical examples are presented to illustrate the obtained results.

1. Introduction

Elastic surface waves have been extensively studied by scientists and engineers because of their practical applicability to disciplines such as seismology, acoustics, geophysics, materials science and nondestructive testing, among others. These waves are of particular importance in seismology, since they cause the most damage and destruction during an earthquake. Hence, elastic surface waves currently receive attention by civil engineers, geologists, and geophysicists interested in seismological applications.

Nondispersive waves propagating along the free surface of an isotropic elastic half-space were first explained by Rayleigh [1] in 1885. Such waves are named Rayleigh waves after him, and the corresponding theory is presented in standard textbooks on elastic waves (e.g. [2,3]). Although the physics of Rayleigh waves is well-known, a mathematical difficulty arises in the analysis: The phase velocity is given as a solution to a nonlinear algebraic equation, known as the secular equation for Rayleigh waves, which cannot be solved in a trivial manner. The usual approach to deal with this equation consists in converting it into a cubic equation and then applying Cardan's formula. However, it is not immediate to determine which of the three solutions corresponds to the desired one. This approach was followed by Hayes and Rivlin [4] to study the nature of the solutions, and by Rahman and Barber [5], Malischewsky [6,7] and Pham and Ogden [8] to obtain exact analytical formulae for the Rayleigh wave velocity. Approximate but simpler formulae were provided by Rahman and Michelitsch [9] and Li [10]. Nkemzi [11] followed a different approach, obtaining a formula for the Rayleigh-wave velocity using Riemann problem theory. However, this formula is rather unwieldy and the final result is wrong (see [6]).

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The extension of surface wave theory to non-isotropic elastic solids is not a trivial matter and it has been the subject of many studies. Buchwald characterised the Rayleigh waves existing in transversely isotropic media [12] and anisotropic media [13] from a theoretical point of view. The case of an orthotropic medium was studied by Stoneley [14], including an explicit secular equation for the first time, and more recently by Pham and Ogden [15], giving an explicit formula for the Rayleigh-wave velocity. Ting [16] provided an explicit secular equation for an elastic solid of general anisotropy.

In the context of Rayleigh waves, it is assumed that the traction, i.e. the local force per area unit, vanishes on the surface (as in all the works mentioned above). From a mathematical point of view, a traction-free surface is described by Neumann boundary conditions. Other type of boundary conditions are rarely considered in geophysics or seismology. In other fields of physics, such as acoustics and electromagnetism, it is common to use impedance (or Robin) boundary conditions, that is, when a linear combination of the unknown function and their derivatives is prescribed on the boundary. In the framework of elasticity theory, Tiersten [17] obtained impedance-like boundary conditions to simulate the effect of a thin layer of a different material over an elastic half-space, using the approximate equations of low-frequency extension and flexure of thin plates. These boundary conditions specify traction in terms of displacement and its derivatives. The monograph by Malischewsky [18] analysed Tiersten's boundary conditions, providing a secular equation for Rayleigh waves. Indeed, they were applied extensively in this work to study the passage of seismic surface waves through discontinuities. An improvement to Tiersten's boundary conditions was proposed by Bövik [19], using a perturbation technique where all terms being linear in the layer thickness are taken into account. These boundary conditions look like those by Tiersten plus some additional terms that correspond to derivatives of the traction. In a very recent article by Malischewsky [20], the potential applicability of impedance boundary conditions to propagation of seismic waves was discussed. The results by Tiersten [17] and Bövik [19] were mentioned as ones of the most important boundary conditions of this type.

The present work considers an isotropic elastic half-space with particular impedance boundary conditions prescribed on the surface, where the normal traction vanishes and the shear traction is proportional to the tangential displacement times the frequency. We perform a detailed analysis of the surface waves that appear with these boundary conditions. For this, the corresponding secular equation is obtained explicitly and the nature of its solutions is analysed mathematically in terms of the impedance parameter. These impedance boundary conditions have some similar features with certain transmission conditions that describe a loosely-bonded interface between two elastic solids (cf. [21,22]). In this kind of models, the shear traction is assumed to be proportional to the displacement or the velocity parallel to the interface, allowing a finite amount of slip between the two contacting surfaces. Transmission conditions of this kind were also treated by Malischewsky, referred to as the boundary conditions after Pod''yapol'sky [20]. In acoustics, impedance boundary conditions are interesting since, unlike the usual Dirichlet or Neumann boundary conditions, they allow in some cases the existence of Rayleigh-type surface waves, depending on the impedance parameter. This fact was shown by Durán et al., considering the Helmholtz equation in a half-plane [23] and in a half-space [24]. The same impedance boundary conditions used herein were already considered by the authors in the context of an effective computation of a Green's function in the elastic half-plane [25].

2. Basic equations

Given Cartesian coordinates (x_1, x_2, x_3), we consider the half-space $x_2 > 0$ occupied by an isotropic elastic medium with constant mass density ρ. We assume a two-dimensional motion in the (x_1, x_2) plane, such that the displacement components (u_1, u_2, u_3) satisfy $u_i = u_i(x_1, x_2, t)$ for $1 \le i \le 2$ and $u_3 \equiv 0$, where t is the time. The relevant components of the stress tensor are denoted by σ_{ij}, where $1 \le i, j \le 2$. They are given in terms of the displacement through the Hooke's law for isotropic media:

$$\sigma_{ij} = \lambda u_{k,k} + \mu(u_{i,j} + u_{j,i}), \tag{1}$$

where $\lambda,\ \mu$ are the Lamé's constants. The commas denote partial differentiation with respect to the spatial variables and the repeated index k implies summation. From now on, we assume a harmonic dependence on time of the form $e^{-\mathrm{i}\omega t}$, where ω is the angular frequency. The plane motion is then governed by the equation

$$\sigma_{ij,j} + \rho\omega^2 u_i = 0 \quad 1 \le i \le 2. \tag{2}$$

In order to solve this equation, we use the Helmholtz theorem, which allows us to express the displacement vector in terms of a scalar potential and a vector potential (cf. [2,3]). In a two-dimensional motion, only one component of the vector potential has relevance, so actually it is possible to deal with two scalar potentials. Therefore, u_1 and u_2 are expressed as

$$u_1 = \varphi_{,1} + \psi_{,2}, \qquad u_2 = \varphi_{,2} - \psi_{,1}, \tag{3}$$

where $\varphi,\ \psi$ denote the scalar potentials. Replacing (3) in (2) and combining with (1), we obtain that a sufficient condition so as to fulfil (2) is that φ and ψ satisfy separately two Helmholtz equations

$$\Delta\varphi + k_L^2\,\varphi = 0, \qquad \Delta\psi + k_T^2\,\psi = 0, \tag{4}$$

where Δ denotes the Laplacian and $k_L = \omega\sqrt{\rho/(\lambda + 2\mu)}$, $k_T = \omega\sqrt{\rho/\mu}$ are the wave numbers associated with the longitudinal (pressure) and transverse (shear) elastic waves, respectively. They satisfy $2k_L^2 < k_T^2$.

3. Impedance boundary conditions and the secular equation

3.1. Impedance boundary conditions

The impedance boundary conditions proposed by Tiersten [17] were expressed in a convenient form by Malischewsky [18,20] in terms of stresses and displacements. In the two-dimensional case, these boundary conditions take the form (summation convention does not hold for underlined indices)

$$\sigma_{i2} + \varepsilon_{\underline{i}} u_{\underline{i}} = 0, \quad \text{for } x_2 = 0, \tag{5}$$

where the impedance parameters ε_i have the dimensions of stress/length. In the context of Tiersten's theory, these parameters have specific values as functions of the thickness of the thin layer, the elastic parameters, and the frequency. If we allow the parameters ε_i to take any value, relation (5) can be regarded as a general form of impedance boundary conditions in elasticity. In particular, if $\varepsilon_i = 0$, we retrieve the well-known traction-free boundary conditions. In the present work, we assume that the surface of the half-space is free of normal traction, and the shear traction depends linearly on the tangential displacement times the frequency. We thus assume that $\varepsilon_1 = \omega Z$ and $\varepsilon_2 = 0$, where Z is an impedance parameter that has the dimensions of stress/velocity. The impedance boundary conditions are then expressed as

$$\sigma_{12} + \omega Z u_1 = 0, \qquad \sigma_{22} = 0, \quad \text{for } x_2 = 0. \tag{6}$$

The impedance parameter Z is assumed strictly real. In particular, setting $Z = 0$ leads to traction-free boundary conditions. On the other hand, the limit $|Z| \to +\infty$ is equivalent to a vanishing tangential displacement.

3.2. The secular equation

Next, we seek solutions to (2) and (6) that correspond to surface waves. The procedure to determine such solutions is mainly based upon that made by Achenbach [2] and Harris [3] for a traction-free boundary. A surface wave has oscillatory behaviour along the surface and decays exponentially away from the boundary. The potentials φ and ψ are then written as

$$\varphi(x_1, x_2) = A_L\, e^{\mathrm{i}kx_1} e^{-k\gamma_L x_2}, \qquad \psi(x_1, x_2) = A_T\, e^{\mathrm{i}kx_1} e^{-k\gamma_T x_2}, \tag{7}$$

where A_L and A_T are arbitrary amplitudes, k is an unknown wave number associated with the surface wave, and the quantities $\gamma_L,\ \gamma_T$ have to be chosen so that both Helmholtz equations (4) hold. Replacing (7) in (4), we obtain that k is related to γ_L and γ_T through the expression

$$k\gamma_\alpha = \sqrt{k^2 - k_\alpha^2}, \quad \alpha = L,\ T. \tag{8}$$

It should be observed that the square root $\sqrt{k^2 - k_\alpha^2}$ takes complex values when $k < k_\alpha$, so it is necessary to give it an exact meaning as a complex-valued function. For this, we express it as the product between $\sqrt{k - k_\alpha}$ and $\sqrt{k + k_\alpha}$, and each one of these two square roots are defined using the following analytic branches of the logarithm on the complex plane, respectively:

$$D^+ = \mathbb{C} \setminus \{k :\ \mathfrak{Re}(k) = k_\alpha,\ \mathfrak{Im}(k) \geq 0\}, \tag{9a}$$

$$D^- = \mathbb{C} \setminus \{k :\ \mathfrak{Re}(k) = -k_\alpha,\ \mathfrak{Im}(k) \leq 0\}, \tag{9b}$$

obtaining that the complex map $k \longrightarrow \sqrt{k^2 - k_\alpha^2}$ is analytic in the region $D^+ \cap D^-$. In addition, it is possible to show that an exact expression for the square root defined like that is

$$\sqrt{k^2 - k_\alpha^2} = -\mathrm{i}k_\alpha \, \exp\left(\int_0^k \frac{\eta}{\eta^2 - k^2}\, \mathrm{d}\eta\right). \tag{10}$$

This definition implies that the square root has positive real part when $k > k_\alpha$ (For more details about this, see [23,24]). The displacement components are obtained by substitution of (7) in (3), yielding

$$u_1(x_1, x_2) = e^{\mathrm{i}kx_1}\left(-\mathrm{i}A_L e^{-k\gamma_L x_2} + A_T\gamma_T e^{-k\gamma_T x_2}\right), \tag{11a}$$

$$u_2(x_1, x_2) = e^{\mathrm{i}kx_1}\left(A_L\gamma_L e^{-k\gamma_L x_2} + \mathrm{i}A_T e^{-k\gamma_T x_2}\right). \tag{11b}$$

Replacing (11) in (6) and expanding, we obtain that in order to fulfil the impedance boundary conditions, it is required that the amplitudes satisfy the homogeneous linear system of equations given by

$$\mathrm{i}\,(2k\mu\gamma_L - \omega Z)\, A_L - \left(k\mu\left(1 + \gamma_T^2\right) - \omega Z\gamma_T\right) A_T = 0, \tag{12a}$$

$$k\mu\left(1 + \gamma_T^2\right) A_L + 2\mathrm{i}k\mu\gamma_T A_T = 0. \tag{12b}$$

Therefore, in order to obtain nontrivial solutions to (12), the determinant of the associated matrix is set to zero, arriving at the identity

$$k\mu\left(\left(1+\gamma_T^2\right)^2-4\gamma_L\gamma_T\right)+\omega Z\gamma_T\left(1-\gamma_T^2\right)=0. \tag{13}$$

Substituting $\gamma_L,\ \gamma_T$ from (8) and rearranging, we obtain

$$\left(2k^2-k_T^2\right)^2-4k^2\sqrt{k^2-k_L^2}\sqrt{k^2-k_T^2}+\frac{\omega Z}{\mu}k_T^2\sqrt{k^2-k_T^2}=0, \tag{14}$$

which corresponds to the desired secular equation. This equation can also be expressed in terms of the slowness $s=k/\omega$ (reciprocal of velocity) as

$$\left(2s^2-s_T^2\right)^2-4s^2\sqrt{s^2-s_L^2}\sqrt{s^2-s_T^2}+\frac{Z}{\mu}s_T^2\sqrt{s^2-s_T^2}=0, \tag{15}$$

where $s_L=\sqrt{\rho/(\lambda+2\mu)}$ and $s_T=\sqrt{\rho/\mu}$ are the slownesses of the longitudinal and transverse waves, respectively, which satisfy $2s_L^2<s_T^2$. The advantage of expressing the secular equation in the form (15) is that it does not depend on ω. Given any solution to (15), the associated solution to (14) is simply obtained by multiplying it by ω, so it suffices to deal with (15). The secular equation with Tiersten's boundary conditions (5) was provided by Malischewsky [18]. This equation is expressed in terms of the slowness as

$$\begin{aligned}&\left(2s^2-s_T^2\right)^2-4s^2\sqrt{s^2-s_L^2}\sqrt{s^2-s_T^2}+\frac{1}{\mu\omega}s_T^2\left(\varepsilon_1\sqrt{s^2-s_T^2}+\varepsilon_2\sqrt{s^2-s_L^2}\right)\\&\quad-\frac{1}{\mu^2\omega^2}\varepsilon_1\varepsilon_2\left(s^2-\sqrt{s^2-s_L^2}\sqrt{s^2-s_T^2}\right)=0.\end{aligned} \tag{16}$$

In the traction-free case ($\varepsilon_1=\varepsilon_2=0$), this equation becomes the classic secular equation for Rayleigh waves [2,3], given by

$$\left(2s^2-s_T^2\right)^2-4s^2\sqrt{s^2-s_L^2}\sqrt{s^2-s_T^2}=0. \tag{17}$$

Setting $\varepsilon_1=\omega Z$ and $\varepsilon_2=0$ in (16), it is straightforward to verify that the secular equation given in (15) is retrieved. It is worth remarking that since (15) depends on s through s^2, its solutions necessarily correspond to pairs symmetrically located with respect to the origin in the complex plane. Consequently, we can assume without loss of generality that $\mathfrak{Re}(s)\geq 0$. Nevertheless, only strictly real solutions of (15) are associated with surface waves, since any complex value of s (or actually k) substituted in (7) or (11) would give exponentially increasing amplitudes as $x_1\to+\infty$ or $x_1\to-\infty$, which does not make any physical sense. This fact was demonstrated by Hayes and Rivlin [4] for the traction-free case.

4. Surface wave analysis

4.1. The Rayleigh wave

In the case of a traction-free bounding surface, the existence and uniqueness of the Rayleigh wave is well-known. Its slowness, denoted by s_R, corresponds to the only real solution of (17) in the range $s>s_T$. The existence and uniqueness of this solution was demonstrated by Achenbach [2], using the principle of the argument of complex analysis. In the case of impedance boundary conditions, we establish the existence and uniqueness of the Rayleigh slowness by means of the following proposition.

Proposition 1. *For each impedance $Z\in\mathbb{R}$, the secular equation (15) has one and only one real solution within the range $s>s_T$.*

Proof. Using the fact that (15) is linear in the impedance, we can easily work out its value in terms of s, obtaining a function $Z=Z(s)$, defined as

$$Z(s)=-\frac{\mu}{s_T^2}\,\frac{\left(2s^2-s_T^2\right)^2-4s^2\sqrt{s^2-s_L^2}\sqrt{s^2-s_T^2}}{\sqrt{s^2-s_T^2}}. \tag{18}$$

As $s>s_T>s_L$, both square roots are real, and thus this function takes real values. In what follows, we aim to demonstrate that $Z:(s_T,+\infty)\to\mathbb{R}$ is actually a bijection. We begin by proving that it is onto $\mathbb{R}$. It is clear that Z is continuous at any $s>s_T$ and

$$\lim_{s\searrow s_T}Z(s)=-\infty. \tag{19}$$

On the other hand, using Taylor approximations it is possible to obtain an asymptotic expression for the square root as $s \to +\infty$:

$$\sqrt{s^2 - s_\alpha^2} \approx s - \frac{s_\alpha^2}{2s} \quad \alpha = L, T, \tag{20}$$

which is employed to approximate $Z(s)$ for large values of s:

$$Z(s) \approx \frac{2\mu}{s_T^2}\left(s_T^2 - s_L^2\right) s, \tag{21}$$

that is, Z behaves asymptotically as a straight line with positive slope. In particular,

$$\lim_{s \to +\infty} Z(s) = +\infty. \tag{22}$$

Consequently, as Z is a continuous function on the interval $(s_T, +\infty)$ that satisfies (19) and (22), we deduce that it is onto $\mathbb{R}$. Let us prove now that Z is injective. For this, we study the sign of its derivative, which can be expressed as

$$\begin{aligned} Z'(s) = {} & \frac{\mu s}{s_T^2\sqrt{s^2 - s_T^2}} \left\{ \frac{\left(2s^2 - s_T^2\right)^2 - 4s^2\sqrt{s^2 - s_L^2}\sqrt{s^2 - s_T^2}}{s^2 - s_T^2} \right. \\ & \left. + 8\left(\frac{s^2}{2}\left(\frac{\sqrt{s^2 - s_L^2}}{\sqrt{s^2 - s_T^2}} + \frac{\sqrt{s^2 - s_T^2}}{\sqrt{s^2 - s_L^2}} \right) + \sqrt{s^2 - s_L^2}\sqrt{s^2 - s_T^2} - \left(2s^2 - s_T^2\right) \right) \right\}. \end{aligned} \tag{23}$$

Using the Young's inequality $ab \le a^2/2 + b^2/2$, valid for any real numbers a and b, it is immediate that

$$\frac{\sqrt{s^2 - s_L^2}}{\sqrt{s^2 - s_T^2}} + \frac{\sqrt{s^2 - s_T^2}}{\sqrt{s^2 - s_L^2}} \ge 2. \tag{24}$$

Replacing (24) in (23), expanding and regrouping terms, we obtain a lower bound for Z':

$$Z'(s) \ge \frac{\mu s\left(4\left(s^2 - 2s_T^2\right)\sqrt{s^2 - s_T^2}\left(\sqrt{s^2 - s_L^2} - \sqrt{s^2 - s_T^2}\right) + s_T^4\right)}{s_T^2\left(s^2 - s_T^2\right)^{3/2}}. \tag{25}$$

The positiveness of Z' is analysed separately in two cases.

- Case 1: $s_T^2 < s^2 < 2s_T^2$.
  In this case, it holds that $s^2 - 2s_T^2 < 0$. We reexpress (25) as

$$Z'(s) \ge \frac{\mu s\left(s_T^4 + 4\left(2s_T^2 - s^2\right)\left(s^2 - s_T^2 - \sqrt{s^2 - s_L^2}\sqrt{s^2 - s_T^2}\right)\right)}{s_T^2\left(s^2 - s_T^2\right)^{3/2}}, \tag{26}$$

  and using the so-called Young's inequality with ε, that is, $ab \le \varepsilon a^2/2 + b^2/2/\varepsilon$, it is possible to obtain the relation

$$\sqrt{s^2 - s_L^2}\sqrt{s^2 - s_T^2} \le s^2 - s_T^2 + \frac{1}{4}\left(s^2 - s_L^2\right). \tag{27}$$

  Substituting (27) in (26) and expanding yields

$$Z'(s) \ge \frac{\mu s\left(s^4 - (s_L^2 + 2s_T^2)s^2 + (2s_L^2 + s_T^2)s_T^2\right)}{s_T^2\left(s^2 - s_T^2\right)^{3/2}}. \tag{28}$$

  The term between parentheses in (28) corresponds to a parabola in s^2 that opens up. Its minimum value is $s^2 = s_T^2 + s_L^2/2 > 0$, which leads to conclude that $Z'(s) > 0$.
- Case 2: $s^2 \ge 2s_T^2$.
  In this case, we have that $s^2 - 2s_T^2 \ge 0$. Moreover, as $s_L < s_T$, it holds that

$$\sqrt{s^2 - s_L^2} - \sqrt{s^2 - s_T^2} > 0$$

and then we obtain directly from (25) that

$$Z'(s) \geq \frac{\mu s_T^2 s}{\left(s^2 - s_T^2\right)^{3/2}}, \tag{29}$$

which implies that $Z'(s) > 0$.

Therefore, the derivative of Z is positive for all $s \in (s_T, +\infty)$, so Z is an strictly increasing function. In particular, it is an injective function. We conclude that $Z : (s_T, +\infty) \to \mathbb{R}$ is a bijective function, and so is its inverse. Consequently, each impedance $Z \in \mathbb{R}$ has associated one and only one real number in the range $s > s_T$ that is a solution to (15). This solution corresponds to the Rayleigh slowness, denoted by $s_R = s_R(Z)$, which concludes the proof. □

Remark 1. From the proof of Proposition 1, we infer that s_R is a strictly increasing function of Z such that it approaches s_T as $Z \to -\infty$ (see (19)) and it increases linearly as $Z \to +\infty$ (see (21)). In particular, the larger the impedance, the slower the Rayleigh wave propagates along the surface.

Notice that in general it is not evident how to calculate s_R analytically in terms of the other involved variables. Nevertheless, it is quite easy to compute s_R numerically by employing an iterative root-finding algorithm such as the Newton–Raphson method. Indeed, the above analysis provides some basic ideas about where s_R has to be searched, which can be used to select a suitable starting point for the iterations. Hence, the convergence of the method can be achieved within a reasonable number of iterations, with a more than acceptable accuracy.

4.2. An additional surface wave

The existence of solutions to the secular equation (15) in the range $s > s_T$ was well established in the previous subsection. We are now interested in studying possible solutions in the range $s_L < s < s_T$. In the general case $Z \in \mathbb{R}$, determining analytically from the secular equation (15) whether such solutions exist or not is not a simple matter. Therefore, what we do is to solve numerically (15) within this range using the iterative method mentioned above, and the obtained results presented in the next section, will give an answer to this problem. Nevertheless, there is a particular value of the impedance for which a solution to (15) can be analytically determined. This result is established in the next proposition.

Proposition 2. *If the impedance takes the positive value $Z = Z^* \equiv 2\mu\sqrt{s_T^2/2 - s_L^2}$, then the secular equation (15) has one real solution in the range $s_L < s < s_T$. This solution is given by $s = s^* \equiv s_T/\sqrt{2}$.*

Proof. Let us assume that $s_L < s < s_T$. In that case, according to the definition of the complex square roots, we have that

$$\sqrt{s^2 - s_T^2} = -\mathrm{i}\sqrt{s_T^2 - s^2}, \tag{30}$$

and replacing this in (15) yields

$$\left(s_T^2 - 2s^2\right)^2 - \mathrm{i}\left(4s^2\sqrt{s^2 - s_L^2} - \frac{1}{\mu}s_T^2 Z\right)\sqrt{s_T^2 - s^2} = 0. \tag{31}$$

This is a complex identity such that the first term on its left-hand side is real, while the second term is purely imaginary. Applying real part to (31) gives the equation

$$\left(s_T^2 - 2s^2\right)^2 = 0, \tag{32}$$

which is easily solved, yielding the desired slowness:

$$s = s^* = \frac{s_T}{\sqrt{2}}. \tag{33}$$

It is straightforward to verify that $s_L < s^* < s_T$. Substituting $s = s^*$ from (33) in (31) and rearranging gives the equation

$$\frac{s_T^3}{\sqrt{2}}\left(2\sqrt{\frac{s_T^2}{2} - s_L^2} - \frac{1}{\mu}Z\right) = 0, \tag{34}$$

which holds if and only if

$$Z = Z^* = 2\mu\sqrt{\frac{s_T^2}{2} - s_L^2}, \tag{35}$$

which concludes the proof. □

Remark 2. Proposition 2 establishes that there is a certain value of the impedance for which an additional surface wave exists. This surface wave is faster than the transverse wave and slower than the longitudinal wave. An explicit expression for the displacement components can be determined by substituting (33) and (35) in (11) and combining with (12). These components are given, up to a multiplicative constant, by

$$u_1(x_1, x_2) = \rho\, \omega e^{\mathrm{i}k^* x_1}\, e^{-\frac{\omega Z^*}{2\mu} x_2}, \qquad u_2(x_1, x_2) = \mathrm{i}k^* Z^* e^{\mathrm{i}k^* x_1}\, e^{-\frac{\omega Z^*}{2\mu} x_2}, \tag{36}$$

where k^* is the associated wave number, defined as $k^* = \omega s^*$, or equivalently

$$k^* = \frac{k_T}{\sqrt{2}}. \tag{37}$$

Remark 3. Roughly speaking, the impedance Z can be regarded as a measure of coupling between shear traction and tangential displacement on the surface of the half-space. A similar role is played by the bonding parameter in the model of loosely-bonded interface between two half-spaces (cf. [21,22]). This parameter relates the shear traction to the slip on the interface, measuring the degree of bonding between both media. Under this analogy, a particular value of Z as that established by Proposition 2 is then equivalent to a particular degree of bonding. If we extrapolate this idea to the context of the impedance boundary conditions obtained by Tiersten [17,18], we find that the impedance in that case is given in terms of the frequency and the parameters of the thin layer simulated over the half-space, namely its density, its elastic constants and its thickness, so it is not a free parameter as our impedance or the bonding parameter just mentioned. However, the parameters of the thin layer can take different values. Indeed, Tiersten reports different phenomena concerning surface waves that take place, depending on whether the layer loads or stiffens the half-space (cf. [17]), which in practice implies considering varying density and elastic constants. Hence, the impedance in Tiersten's boundary conditions can take particular values in a certain range. If we make the analogy of our boundary conditions with those of Tiersten, a particular value of Z as that stated by Proposition 2 is equivalent to a thin layer with a specific density and elastic constants.

As $s = s^*$ is a real solution of (15) that only exists when $Z = Z^*$, a natural question that arises is how this solution is affected if the value of Z^* is slightly perturbed. A precise answer to this question is given in the next proposition.

Proposition 3. *Let us assume a perturbed value of the impedance*

$$Z = Z^*(1 + \varepsilon), \tag{38}$$

where ε is a real parameter satisfying $|\varepsilon| \ll 1$. Then the secular equation (15) has a complex solution s satisfying $s_L \le \mathfrak{Re}\{s\} \le s_T$. Locally, this solution can be approximated as

$$s = s^* \left(1 + a\,\varepsilon + (b + \mathrm{i}c)\varepsilon^2\right) + o(\varepsilon^3), \tag{39}$$

where a, b and c are real numbers, with $c > 0$.

Proof. In order to simplify the analysis, we deal with s^2 instead of s. Given the parameter ε, we desire to determine (approximately) another parameter $\eta = \eta(\varepsilon) \in \mathbb{C}$ such that the solution s to (15) can be written as

$$s^2 = s^{*2}(1 + \eta). \tag{40}$$

Notice that if $\varepsilon = 0$, then $Z = Z^*$ and consequently $s = s^*$. We thus infer that η must satisfy $\eta(0) = 0$. Replacing (38) and (40) in (15), approximating the square roots by employing second-order Taylor polynomials in η (around $\eta = 0$), and combining with (33) and (35), we obtain that η satisfies the quadratic equation:

$$\left(1 - \frac{1}{2}\,q + \frac{1}{4}\,q^2 - \frac{1}{4}\,\varepsilon + 2\mathrm{i}\sqrt{q}\right)\eta^2 - (2 + q + \varepsilon)\,\eta + 2\varepsilon = 0, \tag{41}$$

where $q = s_T^2/(s_T^2 - 2s_L^2)$. The quadratic equation (41) has two solutions, but only one of them vanishes when $\varepsilon = 0$. We thus choose this solution, which is given by

$$\eta(\varepsilon) = \frac{2 + q + \varepsilon - \left((2+q)^2 - 2\left(2 - 3q + q^2 + 8\mathrm{i}\sqrt{q}\right)\varepsilon + 3\varepsilon^2\right)^{1/2}}{2 - q + \frac{1}{2}\,q^2 - \frac{1}{2}\,\varepsilon + 4\mathrm{i}\sqrt{q}}. \tag{42}$$

As we want to study the behaviour of η for small ε, we approximate (42) by a second-order Taylor expansion, obtaining

$$\eta(\varepsilon) = \frac{2}{2+q}\,\varepsilon - \frac{(4-q)q - 8\mathrm{i}\sqrt{q}}{(2+q)^3}\,\varepsilon^2 + o(\varepsilon^3). \tag{43}$$

Table 1
Mass density, Young's modulus and Poisson's ratio of the materials considered.

Material	ρ (kg/m^3)	E (GPa)	ν
Diabase	2700	90	0.20
Limestone	2400	70	0.30
Gneiss	2800	60	0.24

Hence, replacing (43) in (40) gives

$$s^2 = s^{*2}\left(1 + \frac{2}{2+q}\,\varepsilon - \frac{(4-q)q - 8\mathrm{i}\sqrt{q}}{(2+q)^3}\,\varepsilon^2\right) + o(\varepsilon^3). \tag{44}$$

and applying a square root to both sides of (44), together with another second-order Taylor expansion, we finally obtain

$$s = s^{*}\left(1 + \frac{1}{2+q}\,\varepsilon + \frac{2 + 5q - q^2 + 8\mathrm{i}\sqrt{q}}{2(2+q)^3}\,\varepsilon^2\right) + o(\varepsilon^3). \tag{45}$$

This expression coincides with (39), with

$$a = \frac{1}{2+q}, \qquad b = \frac{2+5q-q^2}{2(2+q)^3}, \qquad c = \frac{4\sqrt{q}}{(2+q)^3}. \tag{46}$$

It is clear that a, b and c are real numbers and $c > 0$, which concludes the proof. □

Remark 4. In virtue of Proposition 3, if the impedance is perturbed around $Z = Z^*$, then the solution $s = s^*$ becomes a complex number with strictly positive imaginary part. Although this complex value of s is a solution of the secular equation (15), it does not have associated a physical behaviour, as mentioned before. If we substitute Z^* and s^* by their perturbed values in (36), we obtain that this expression is exponentially increasing as $x_1 \to -\infty$. Furthermore, this proposition can be interpreted as a local uniqueness result for the additional surface wave. If the value of $Z = Z^*$ is locally perturbed, then this wave is no longer present, so it is unique in a certain interval of values of Z.

5. Numerical results

In this section, the Rayleigh slowness and the additional solution to (15) are computed numerically for different values of the surface impedance Z. We consider three examples of elastic material, namely diabase (volcanic rock), limestone (sedimentary rock) and gneiss (metamorphic rock). A detailed description of these rocks and their characteristics can be found in the handbook by Stacey and Page [26]. Table 1 shows their physical and elastic properties, namely the mass density ρ, the Young's modulus E and the Poisson's ratio ν. The Lamé's constants λ and μ can be obtained in terms of E and ν through the usual formulae

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{47}$$

The numerical values of the longitudinal and the transverse slownesses are shown in Table 2. On the other hand, we consider an impedance Z varying from a minimum value $Z_{\min} = -20$ [MPa s/m] to a maximum value $Z_{\max} = 20$ [MPa s/m]. The Rayleigh slowness is calculated by solving iteratively the secular equation (15), using for this the Newton–Raphson algorithm with a starting point for the iterations located at the region $s > s_T$, as mentioned above. The results are presented in Fig. 1, where we confirm what was stated by Proposition 1: The Rayleigh slowness s_R is an increasing function of the impedance Z, which approaches asymptotically s_T as Z decreases to $-\infty$ and tends to a straight line as Z increases to $+\infty$. The additional solution to (15) is also computed by means of the Newton–Raphson algorithm, but this time the starting point is located within the region $s_L < s < s_T$. The numerical evidence shows that this solution does not exist for all values of Z. In particular, it is not present for Z negative, so the results are presented only for $Z \geq 0$. Figs. 2 and 3 show the real and the imaginary part of the additional solution as a function of the impedance, respectively. Fig. 3 puts in evidence, for each elastic material considered, the existence of a particular value of Z such that the imaginary part of the solution vanishes. This is the value $Z = Z^*$ established by Proposition 2, for which the additional solution becomes real and is given by the slowness $s = s^*$. In addition, Fig. 3 confirms that if we perturb Z around Z^*, then a strictly positive imaginary part appears in the additional solution, as stated by Proposition 3. The numerical values of the constants Z^* and s^* associated with each material are shown in Table 3.

6. Conclusions

The existence of surface waves in an isotropic elastic half-space with impedance boundary conditions is studied. We obtain the explicit secular equation, which has the form of the secular equation of the traction-free case plus an additional

Table 2
Longitudinal and transverse slownesses of the materials considered.

Material	$s_L \times 10^{-4}$ (s/m)	$s_T \times 10^{-4}$ (s/m)
Diabase	1.6432	2.6833
Limestone	1.5959	2.9857
Gneiss	1.9898	3.4020

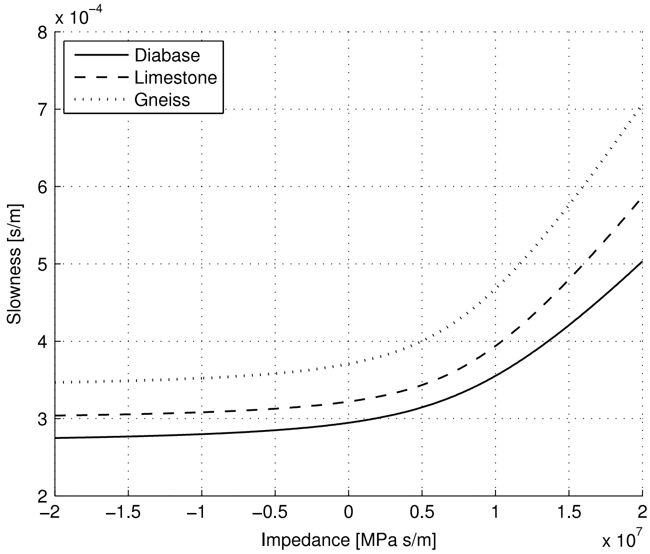

Fig. 1. Rayleigh solution as a function of the impedance.

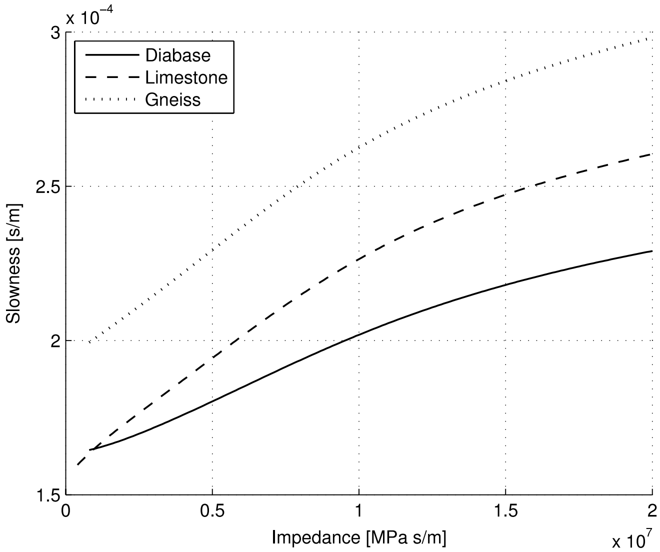

Fig. 2. Real part of the additional solution as a function of the impedance.

Table 3
Impedance Z^* and slowness s^* of the materials considered.

Material	Z^* (MPa s/m)	$s^* \times 10^{-4}$ (s/m)
Diabase	7.1151	1.8974
Limestone	7.4421	2.1112
Gneiss	6.5410	2.4055

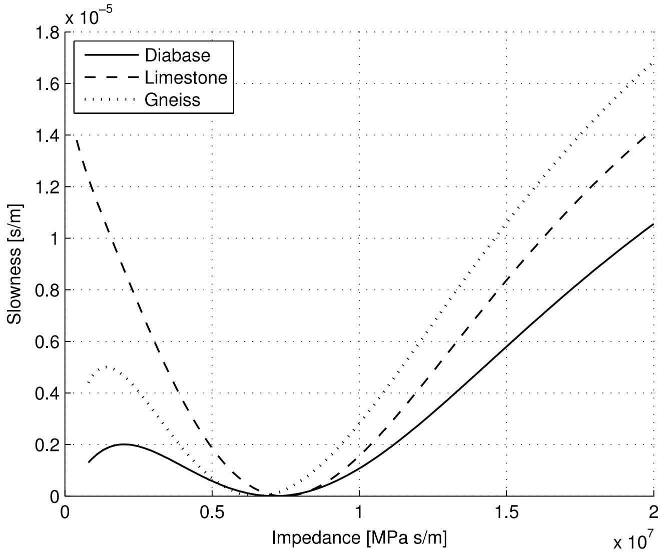

Fig. 3. Imaginary part of the additional solution as a function of the impedance.

term involving the impedance. We present theoretical results on the existence and uniqueness of surface waves, based on a mathematical analysis of the secular equation. Specifically, it is proven that the Rayleigh wave exists for all values of the impedance and its velocity decreases with the impedance. Furthermore, we prove that an additional surface wave exists when the impedance takes a particular value. The velocity of this wave lies between those of the longitudinal wave and the transverse wave. In addition, a perturbation analysis around this additional solution is performed, showing that it is unique in a certain region. Numerical results for three elastic rock media are presented, which are in agreement with the theoretical results.

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References

[1] L. Rayleigh, On waves propagating along the plane surface of an elastic solid, Proc. R. Soc. Lond. A17 (1885) 4–11.
[2] J.D. Achenbach, Wave Propagation in Elastic Solids, North-Holland, Amsterdam, 1973.
[3] J.G. Harris, Linear Elastic Waves, Cambridge, New York, 2001.
[4] M. Hayes, R.S. Rivlin, A note on the secular equation for Rayleigh waves, ZAMP 13 (1962) 80–83.
[5] M. Rahman, J.R. Barber, Exact expressions for the roots of the secular equation for Rayleigh waves, ASME J. Appl. Mech. 62 (1995) 250–252.
[6] P.G. Malischewsky, Comment to A new formula for the velocity of Rayleigh waves by D. Nkemzi [Wave Motion 26 (1997) 199-205], Wave Motion 31 (2000) 93–96.
[7] P.G. Malischewsky, A note on Rayleigh-wave velocities as a function of the material parameters, Geofis. Int. 43 (2004) 507–509.
[8] Pham Chi Vinh, R.W. Ogden, On formulas for the Rayleigh wave speed, Wave Motion 39 (2004) 191–197.
[9] M. Rahman, T. Michelitsch, A note on the formula for the Rayleigh wave speed, Wave Motion 43 (2006) 272–276.
[10] X.-F. Li, On approximate analytic expressions for the velocity of Rayleigh waves, Wave Motion 44 (2006) 120–127.
[11] D. Nkemzi, A new formula for the velocity of Rayleigh waves, Wave Motion 26 (1997) 199–205.
[12] V.T. Buchwald, Rayleigh waves in transversely isotropic media, Quart. J. Mech. Appl. Math. 14 (1961) 293–317.
[13] V.T. Buchwald, Rayleigh waves in anisotropic media, Quart. J. Mech. Appl. Math. 14 (1961) 461–469.
[14] R. Stoneley, The propagation of surface waves in an elastic medium with orthorhombic symmetry, Geophys. J. Astron. Soc 8 (1963) 176–186.
[15] Pham Chi Vinh, R.W. Ogden, Formulas for the Rayleigh wave speed in orthotropic elastic solids, Arch. Mech. 56 (2004) 247–265.
[16] T.C.T. Ting, An explicit secular equation for surface waves in an elastic material of general anisotropy, Quart. J. Mech. Appl. Math. 55 (2002) 297–311.
[17] H.F. Tiersten, Elastic surface waves guided by thin films, J. Appl. Phys. 40 (1969) 770–789.
[18] P.G. Malischewsky, Surface Waves and Discontinuities, Elsevier, Amsterdam, 1987.
[19] P. Bövik, A comparison between the Tiersten model and $O(h)$ boundary conditions for elastic surface waves guided by thin layers, Trans. ASME 63 (1996) 162–167.
[20] P.G. Malischewsky, Seismological implications of impedance-like boundary conditions, in: Proceedings of Days on Diffraction, St. Petersburg/Russia, 2011, pp. 131–134.
[21] G.S. Murty, Reflection, transmission and attenuation of elastic waves at a loosely-bonded interface of two half spaces, Geophys. J. Roy. Astr. S. 44 (1976) 389–404.
[22] M. Durán, J.-C. Nédélec, Un problème spectral issu d'un couplage élasto-acoustique, ESAIM: M2AN 34 (2000) 835–857.
[23] M. Durán, I. Muga, J.-C. Nédélec, The Helmholtz equation in a locally perturbed half-plane with passive boundary, IMA J. Appl. Math. 71 (2006) 853–876.
[24] M. Durán, I. Muga, J.-C. Nédélec, The Helmholtz equation in a locally perturbed half-space with non-absorbing boundary, Arch. Ration. Mech. Anal. 191 (2009) 143–172.
[25] M. Durán, E. Godoy, J.-C. Nédélec, Theoretical aspects and numerical computation of the time-harmonic Green's function for an isotropic elastic half-plane with an impedance boundary condition, ESAIM: M2AN 44 (2010) 671–692.
[26] T.R. Stacey, C.H. Page, Practical Handbook for Underground Rock Mechanics, Trans Tech Publications, 1986.